On some quantum bounded symmetric domains

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February 2, 2008

Abstract

In the framework of quantum group theory we obtain a noncommutative analog for the algebra of functions in a bounded symmetric domain, endowed with a whole symmetry. Also we provide a construction for its faithfull irreducible representation and an invariant integral over the bounded symmetric domain.

1 Introduction

Recall some well-known facts on bounded symmetric domains. We focus on a series of bounded symmetric domains $\mathbb{D}$ from the well-known Cartan list.

Let

$$\mathfrak{a} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ is isomorphic to the Lie algebra with generators $e_i, f_i, h_i, i = 1, ..., n$ and relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$

$$[e_i, f_j] = \delta_{ij} h_i \quad i, j = 1, ..., n,$$

together with Serre's relations. The linear span $\mathfrak{h}$ of $h_1, h_2, \dots, h_n$ is a Cartan subalgebra, and the linear functionals $\alpha_1, \alpha_2, \dots, \alpha_n$ on $\mathfrak{h}$ defined by

$$\alpha_j(h_i) = a_{ij}, \qquad i, j = 1, 2, \dots, n$$

form a system of simple roots for the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let $h_0 \in \mathfrak{h}$ be the element given by

$$\alpha_n(h_0) = 2, \qquad \alpha_j(h_0) = 0,\ j < n.$$

It is easy to prove that $h_0 = h_1 + 2h_2 + ... + nh_n$.

Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie subalgebra generated by

$$e_i,\ f_i, \quad i \neq n, \qquad h_i,\ i = 1, ..., n.$$

The pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian symmetric, i.e. $\mathfrak{g}$ is equipped with a grading $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$, where

$$\mathfrak{p}^\pm = \{\xi \in \mathfrak{g} |\ [h_0, \xi] = \pm 2\xi\},$$
$$\mathfrak{k} = \{\xi \in \mathfrak{g} |\ [h_0, \xi] = 0\}.$$

Note that $\mathfrak{p}^-$ is isomorphic to the normed vector space of symmetric complex $n \times n$-matrices with the operator norm. Harish-Chandra proved that an irreducible bounded symmetric domain $\mathbb{D}$ can be embedded into the normed vector space $\mathfrak{p}^-$ as the unit ball.

In this paper we consider quantum analogs for the algebra $\mathbb{C}[\mathfrak{p}^-]$ of holomorphic polynomials on $\mathfrak{p}^-$ and the algebra $\mathrm{Pol}(\mathfrak{p}^-)$ of polynomials on $\mathfrak{p}^-_{\mathbb{R}}$ and give some results on representation theory on $\mathbb{D}$.

2 Algebras $\mathbb{C}[\mathfrak{p}^-]_q$ and $\mathrm{Pol}(\mathfrak{p}^-)_q$

In the sequel $q \in (0, 1)$, $\mathbb{C}$ is the ground field, and all the algebras are assumed associative and unital.

Let $d_i, i = 1, ..., n$ be coprime numbers that symmetrize the Cartan matrix $\mathfrak{a}$. One can check that $d_i = 1$ for $i = 1, ..., n-1$ and $d_n = 2$.

Denote by $U_q\mathfrak{g} = U_q\mathfrak{sp}_{2n}$ a Hopf algebra with generators K_i, K_i^{-1}, E_i, F_i, $i = 1, \ldots, n$, and relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j = q^{d_i a_{ij}} E_j K_i, \quad K_i F_j = q^{-d_i a_{ij}} F_j K_i, \quad i, j = 1, ..., n$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},$$

together with q-analogs of the well-known Serre relations [4].

The coproduct Δ, the counit ε and the antipod S are defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,\ \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

Denote by $U_q\mathfrak{k} \subset U_q\mathfrak{g}$ a Hopf subalgebra generated by

$$E_j,\ F_j, \quad j < n \qquad \text{and} \qquad K_i^{\pm 1}, \quad i = 1, ..., n.$$

$U_q\mathfrak{g}$ can be equipped with the involution $*$ given by

$$(K_j^\pm)^* = K_j^\pm, \quad j = 1, ..., n.$$

$$E_j^* = \begin{cases} K_j F_j, & j < n, \\ -K_j F_j, & j = n, \end{cases} \quad F_j^* = \begin{cases} E_j K_j^{-1}, & j < n, \\ -E_j K_j^{-1}, & j = n. \end{cases}$$

The $*$-Hopf algebra $(U_q\mathfrak{g}, *)$ is a quantum analog for the universal enveloping algebra of $\mathfrak{sp}_{2n}(\mathbb{R})$.

Recall a general notion [4]. Let F be an algebra which is also a module over a Hopf algebra A. F is called an A-module algebra if the multiplication $m : F \otimes F \to F$ is a morphism of A-modules and the unit $1 \in F$ is A-invariant.

In [1], L.Vaksman with his collaborators introduced a $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathfrak{p}^-]_q$, a quantum analog of the algebra of holomorphic polynomials on $\mathfrak{p}^-$. They followed V.Drinfeld's approach to producing quantum analogs of function algebras by duality. The next two theorems give an explicit description of the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathfrak{p}^-]_q$ in terms of generators and relations.

Theorem 1 *$\mathbb{C}[\mathfrak{p}^-]_q$ is isomorphic to the algebra generated by $z_{ij}, 1 \le j \le i \le n$, whose ddefinig relations are the following:*

$$z_{ij}z_{kl} = q^2 z_{kl}z_{ij}, \qquad i = j = l < k \tag{1}$$
$$z_{ij}z_{kl} = q^2 z_{kl}z_{ij}, \qquad j < i = k = l \tag{2}$$
$$z_{ij}z_{kl} = q z_{kl}z_{ij}, \qquad j < l < i = k \tag{3}$$
$$z_{ij}z_{kl} = q z_{kl}z_{ij}, \qquad j = l < i < k \tag{4}$$
$$z_{kl}z_{ij} = z_{ij}z_{kl}, \qquad j < l \le k < i \tag{5}$$
$$z_{ij}z_{kl} = z_{kl}z_{ij} + q(q^2 - q^{-2})z_{lj}z_{ki}, \qquad i = j < k = l \tag{6}$$
$$z_{ij}z_{kl} = z_{kl}z_{ij} + (q^2 - q^{-2})z_{lj}z_{ki}, \qquad i = j < l < k \tag{7}$$
$$z_{ij}z_{kl} = z_{kl}z_{ij} + (q^2 - q^{-2})z_{lj}z_{ki}, \qquad j < i < k = l \tag{8}$$
$$z_{ij}z_{kl} = z_{kl}z_{ij} + (q - q^{-1})(qz_{li}z_{kj} + z_{ki}z_{lj}), \qquad j < i < l < k \tag{9}$$
$$z_{ij}z_{kl} = z_{kl}z_{ij} + (q - q^{-1})z_{il}z_{kj}, \qquad j < l < i < k \tag{10}$$
$$z_{ij}z_{kl} = qz_{kl}z_{ij} + (q - q^{-1})z_{il}z_{kj}, \qquad j < i = l < k. \tag{11}$$

Theorem 2 *$\mathbb{C}[\mathfrak{p}^-]_q$ is equipped with the structure of $U_q\mathfrak{g}$-module algebra defined on generators as follows: for $k < n$*

$$K_k z_{ij} = \begin{cases} q^2 z_{ij}, & i = j = k, \\ q^{-2} z_{ij}, & i = j = k+1, \\ q z_{ij}, & i = k > j \ \textit{ or } \ i-1 > k = j, \\ q^{-1} z_{ij}, & i-1 = k > j \ \textit{ or } \ i > k+1 = j, \\ z_{ij}, & \textit{otherwise.} \end{cases}$$

$$E_k z_{ij} = q^{-1/2} \begin{cases} (q + q^{-1}) z_{i\,j-1}, & i = j = k+1, \\ z_{i-1\,j}, & i = k+1 > j, \\ z_{i\,j-1}, & i > k+1 = j, \\ 0, & \textit{otherwise,} \end{cases} \qquad F_k z_{ij} = q^{1/2} \begin{cases} (q + q^{-1}) z_{i+1\,j}, & i = j = k, \\ z_{i+1\,j}, & i = k > j, \\ z_{i\,j+1}, & i > k = j, \\ 0, & \textit{otherwise.} \end{cases}$$

and

$$K_n z_{ij} = \begin{cases} q^4 z_{ij}, & i = j = n, \\ q^2 z_{ij}, & i = n > j, \\ z_{ij}, & \textit{otherwise.} \end{cases}$$

$$F_n z_{ij} = \begin{cases} q, & i = j = n, \\ 0, & \textit{otherwise.} \end{cases} \qquad E_n z_{ij} = -\begin{cases} q z_{nn} z_{ij}, & i = n \ge j, \\ z_{ni} z_{nj}, & \textit{otherwise.} \end{cases}$$

Note, that the algebra $\mathbb{C}[\mathfrak{p}^-]_q$ was obtained in the Kamita's paper [3], but only as a $U_q\mathfrak{k}$-module algebra.

In [1], a $(U_q\mathfrak{g},*)$-module $*$-algebra $\mathrm{Pol}(\mathfrak{p}^-)_q$, (i.e. we have $(\xi f)^* = S(\xi)^* f^*$ for all $\xi \in U_q\mathfrak{g}, f \in \mathrm{Pol}(\mathfrak{p}^-)_q$) was introduced. It is considered as a quantum analog of the algebra of polynomials on $\mathfrak{p}^-_{\mathbb{R}}$. Also the existence and uniqueness of the faithfull irreducible $*$-representation of $\mathrm{Pol}(\mathfrak{p}^-)_q$ and the $(U_q\mathfrak{g},*)$-invariant integral over the bounded symmetric domain $\mathbb{D}$ were proved (see [2, chap.2]).

Proposition 1 *A list of defining relations for the $*$-algebra $\mathrm{Pol}(\mathfrak{p}^-)_q$ consists of the relations (1)-(11) and*

$$z_{ij}^* z_{kl} = z_{kl} z_{ij}^*, \qquad j \neq k,l \ \&\ i \neq k,l,$$

$$z_{ij}^* z_{kl} = q z_{kl} z_{ij}^* - (q^{-1}-q)\sum_{m>k} z_{ml} z_{mj}^*, \qquad i=k>j>l,$$

$$z_{ij}^* z_{kl} = q z_{kl} z_{ij}^* - q(q^{-1}-q)\Big(\sum_{i\geq m>k} z_{ml} z_{im}^* + q\sum_{m>i} z_{ml} z_{mi}^*\Big), \qquad i>k=j>l,$$

$$z_{ij}^* z_{kl} = q z_{kl} z_{ij}^* - (q^{-1}-q)\Big(\sum_{k\geq m>l} z_{km} z_{im}^* + q\sum_{i\geq m>k} z_{mk} z_{im}^* + q^2\sum_{m>i} z_{mk} z_{mi}^*\Big), \qquad i>k>j=l,$$

$$z_{ij}^* z_{kl} = q^2 z_{kl} z_{ij}^* - (1+q^2)(q^{-1}-q)\Big(\sum_{i\geq m>l} z_{mk} z_{im}^* + q\sum_{m>i} z_{mk} z_{mi}^*\Big), \qquad i>j=k=l,$$

$$z_{ij}^* z_{kl} = q^2 z_{kl} z_{ij}^* - (1+q^2)(q^{-1}-q)\sum_{m>k} z_{ml} z_{mi}^*, \qquad i=j=k>l,$$

$$z_{ij}^* z_{kl} = q^2 z_{kl} z_{ij}^* - q(q^{-1}-q)\Big(\sum_{i\geq k'>j} z_{kk'} z_{ik'}^* + \sum_{k'>i} z_{k'l} z_{k'j}^* + q^2\sum_{k'>i} z_{k'k} z_{k'i}^*\Big)$$
$$+ (q^{-1}-q)^2 \sum_{k'>i,l'>j,k'>l'} z_{k'l'} z_{k'l'}^* + (q^{-1}-q)^2\sum_{k'>i} z_{k'k'} z_{k'k'}^* + 1 - q^2, \qquad i=k>j=l,$$

$$z_{ij}^* z_{kl} = q^4 z_{kl} z_{ij}^* - q(q^{-1}-q)(1+q^2)^2\sum_{k'>i} z_{k'l} z_{k'j}^*$$
$$+ (q^{-1}-q)^2(1+q^2)\sum_{k'>i} z_{k'k'} z_{k'k'}^* + (q^{-1}-q)^2(1+q^2)^2 \sum_{k'>j'>i} z_{k'j'} z_{k'j'}^* + 1 - q^4, \qquad i=j=k=l.$$

together with relations which can be obtained from the above due to obvious involution properties (the involution $*$ is defined naturally $*: z_{ij} \mapsto z_{ij}^*$).

3 The faithfull representation and the invariant integral

In this section we outline some results which were obtained in [2, chap.2].

Introduce an irreducible $*$-representation of $\mathrm{Pol}(\mathfrak{p}^-)_q$. Let $\mathcal{H}$ be a $\mathrm{Pol}(\mathfrak{p}^-)_q$-module with a single generator v_0 and the relations

$$z_{ij}^* v_0 = 0, \quad 1 \leq j \leq i \leq n.$$

Proposition 2 *[2, sect. 2.2] 1. $\mathcal{H} = \mathbb{C}[\mathfrak{p}^-]_q v_0$.*

2. There exists a unique sesquilinear form $(\cdot,\cdot)$ on $\mathcal{H}$ with the properties: i) $(v_0, v_0) = 1$; ii) $(fv, w) = (v, f^* w)$ for all $v, w \in \mathcal{H}$, $f \in \mathrm{Pol}(\mathfrak{p}^-)_q$.
3. The form $(\cdot,\cdot)$ is positive definite on $\mathcal{H}$.

Denote by T_F the representation of $\mathrm{Pol}(\mathfrak{p}^-)_q$:

$$T_F(f)v = fv, \quad f \in \mathrm{Pol}(\mathfrak{p}^-)_q, v \in \mathcal{H}.$$

Theorem 3 *[2, sect. 2.2] 1. T_F is a faithful irreducible $*$-representation.*
2. A $\mathrm{Pol}(\mathfrak{p}^-)_q$-representation with such properties is unique up to unitary equivalence.

Let $d\nu$ be an invariant measure on the irreducible bounded symmetric domain $\mathbb{D}$. Note that

$$\int_{\mathbb{D}} f d\nu = \infty, \quad f \in \mathrm{Pol}(\mathfrak{p}^-), f \neq 0.$$

So we have to construct a quantum analog of the algebra of smooth functions on $\mathbb{D}$ with compact supports and to define an invariant integral on it.

The algebra $\mathbb{C}[\mathfrak{p}^-]_q$ is equipped with a natural grading

$$\mathbb{C}[\mathfrak{p}^-]_q = \bigoplus_{k=0}^{\infty} \mathbb{C}[\mathfrak{p}^-]_{q,k}, \quad \deg z_{ij} = 1.$$

Extend the $*$-algebra $\mathrm{Pol}(\mathfrak{p}^-)_q$ by attaching an element f_0 which satisfies the following relations:

$$f_0^2 = f_0,\ f_0^* = f_0, \quad \psi^* f_0 = f_0 \psi = 0, \quad \psi \in \mathbb{C}[\mathfrak{p}^-]_{q,1}.$$

The two-sided ideal of the extended algebra

$$\mathcal{D}(\mathbb{D})_q \stackrel{\mathrm{def}}{=} \mathrm{Pol}(\mathfrak{p}^-)_q \cdot f_0 \cdot \mathrm{Pol}(\mathfrak{p}^-)_q$$

is treated as a quantum analog of the space of smooth functions with compact supports on $\mathbb{D}$.

$\mathcal{D}(\mathbb{D})_q$ is equipped with a $(U_q\mathfrak{g}, *)$-module algebra structure via

$$F_j f_0 = \begin{cases} -\frac{q^5}{1-q^4} f_0 z_{nn}^*, & j = n, \\ 0, & j \neq n, \end{cases} \qquad E_j f_0 = \begin{cases} -\frac{q}{1-q^4} z_{nn} f_0, & j = n, \\ 0, & j \neq n. \end{cases}$$

Let $\mathcal{H}_F \stackrel{\mathrm{def}}{=} \mathbb{C}[\mathfrak{p}^-]_q f_0$.

$\mathcal{H}_F$ is a $\mathcal{D}(\mathbb{D})_q$-module, a $\mathrm{Pol}(\mathfrak{p}^-)_q$-module, and a $U_q\mathfrak{g}$-module. Denote by $\mathcal{T}_F$ the corresponding representations of $\mathcal{D}(\mathbb{D})_q$ and $\mathrm{Pol}(\mathfrak{p}^-)_q$ in the vector space $\mathcal{H}_F$ (these representations are related with their faithful irreducible $*$-representations) and by Γ the representation of $U_q\mathfrak{g}$.

Define a linear functional on $\mathcal{D}(\mathbb{D})_q$:

$$\int_{\mathbb{D}_q} f d\nu = (1-q^4)^{\frac{n(n+1)}{2}} \mathrm{tr}(\mathcal{T}_F(f)\Gamma(K^{-1})), \quad f \in \mathcal{D}(\mathbb{D})_q,$$

with $K = K_1^{2n} K_2^{2(2n-1)} ... K_{n-1}^{(n-1)(n+2)} K_n^{n(n+1)/2}$.

Theorem 4 *[2, sect. 2.2] 1. The above integral is $U_q\mathfrak{g}$-invariant and positive, i.e.*

$$\int_{\mathbb{D}_q} (\xi \cdot f) d\nu = \varepsilon(\xi) \int_{\mathbb{D}_q} f d\nu, \quad \xi \in U_q\mathfrak{g}, f \in \mathcal{D}(\mathbb{D})_q,$$

and

$$\int_{\mathbb{D}_q} (f^* f) d\nu > 0, \quad f \in \mathcal{D}(\mathbb{D})_q, f \neq 0.$$

2. A positive $U_q\mathfrak{g}$-invariant integral on $\mathcal{D}(\mathbb{D})_q$ is unique up to a constant multiple.

Acknowledgment

The author would like to thank professor L.Vaksman for statements of the problems and helpful discussions.

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